HARTOGS'S PHENOMENON

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1. INTRODUCTION

In calculus, one learns about four elementary kinds of discontinuities an otherwise differentiable real function $f(x)$ can have. First are *removable discontinuities*, which are points p at which f is undefined, yet the limit $\lim_{x\to p} f(x)$ exists. For instance, the function

$$f(x) = \frac{(x-1)(x+1)}{x-1}$$

satisfies $\lim_{x\to 1} f(x) = 2$, yet naively plugging in 1 yields the indeterminate form $0/0$.

Another common type of discontinuity is a *vertical asymptote*, which is a point p at which $\lim_{x\to p} f(x) = \pm\infty$. There are many standard examples of such functions: for instance, $f(x) = 1/(x-3)$ has a vertical asymptote at 3 since its denominator vanishes.

Finally, there are *jump discontinuities*, or points at which the left and right-sided limits exist but are different, and *essential discontinuities*, or points at which the limit does not exist at all.

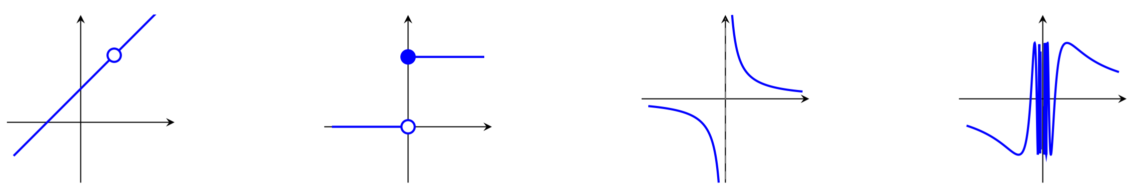

FIGURE 1. (a) Removable discontinuity of $\frac{x^2-1}{x-1}$ at $x = 1$, (b) Jump discontinuity at $x = 0$, (c) Vertical asymptote of $\frac{1}{x}$ at $x = 0$, and (d) Essential discontinuity of $\sin(\frac{1}{x})$ at $x = 0$.

There is a similar classification for meromorphic[1] functions $f(z)$ of one complex variable. Here, f is said to have a *removable singularity* at p if $\lim_{z\to p} f(z)$ exists and is finite, and $f(z)$ is said to have a *pole* at p if $\lim_{z\to p} f(z) = \infty$. Note that while the latter has a different name, it is identical to the definition of a vertical asymptote. Finally, if $\lim_{z\to p} f(z)$ does not exist, p is called an *essential singularity* of f. One important distinction is that in the complex case, there is no such thing as a jump discontinuity: since $\mathbb{C}$ is two-dimensional, there are infinitely many ways to approach a given point p, not just left and right.

[1] All of the requisite terminology will be established in Section 2.

What happens when we generalize this classification of isolated singularities to multiple variables? In the real case, the classification is relatively uninteresting. Here, we too have three kinds of discontinuities: removable, essential, and infinite, with jump discontinuities being omitted for the same reason as the single-variable complex case.

In the multivariable complex case, however, something very special happens. As it turns out, all isolated singularities of a meromorphic function in two or more variables are removable! More generally, if the set of singularities is compact, it is removable. This amazing fact is known as *Hartogs's extension theorem*, or Hartogs's phenomenon, and can be stated as follows:

Theorem 1.1 (Hartogs). *Let $K \subset U$ be a compact subset of an open set $U \subseteq \mathbb{C}^n$ such that $U \setminus K$ is connected. If $n \geq 2$, any holomorphic function $f : U \setminus K \to \mathbb{C}$ extends uniquely to a holomorphic function on U.*

In the case $n = 1$, it is easy to find counterexamples to this theorem: for instance, the function $f(z) = 1/z$ has a singularity at $z = 0$, and there is clearly no way to holomorphically extend it to all of $\mathbb{C}$.

The most immediate consequences of Hartogs's phenomenon concern the structure of singularities and zeros.

Corollary 1.2. *Let $n \geq 2$ and let $U \subseteq \mathbb{C}^n$ be open. Then:*

(i) If f is holomorphic on $U \setminus \{p\}$ for some $p \in U$, then f extends holomorphically to all of U.
(ii) Suppose $f : U \to \mathbb{C}$ is holomorphic and not identically zero with a nonempty zero set $Z_f := f^{-1}(0)$. Then, if $U \setminus Z_f$ is connected, then Z_f is not compact.
(iii) If $K \subset \mathbb{C}^n$ is compact, then $\mathbb{C}^n \setminus K$ is not biholomorphic to any bounded domain in $\mathbb{C}^n$.

Each of these corollaries lies in contrast to the corresponding single-variable case. First, (i) provides a multivariable generalization of Riemann's principle for removable singularities, without the condition that f is bounded on U. Second, (ii) shows that in general, the zero sets of holomorphic functions in several variables are not only infinite, but are not even compact. Lastly, (iii) demonstrates a fundamentally different behavior from the single-variable case. Indeed, in the case $n = 1$, $\mathbb{C} \setminus K$ is *always* biholomorphic to a bounded domain, provided that K has a nonempty interior.

Hartogs's phenomenon is one of the most important structure theorems in the theory of several complex variables. It lies at the start of the apparatus of complex geometry, which is today a fundamental tool in both mathematics and theoretical physics.

In Section 2, we will review the theory of complex analysis in one and several variables, and in Section 3, we will prove Hartogs's phenomenon in the case that K is finite.

2. Preliminaries

We begin by reviewing some complex analysis.

2.1. **Functions of One Complex Variable.**

Definition 2.1. Let $U \subseteq \mathbb{C}$ be an open set. A function $f : U \to \mathbb{C}$ is said to be *complex differentiable* at a point $p \in U$ if the limit

$$f'(p) = \lim_{h \to 0} \frac{f(p+h) - f(p)}{h}$$

exists, where $h \in \mathbb{C} \setminus \{0\}$. We say f is *holomorphic* on U if it is complex differentiable at every point of U.

In general, complex differentiability is a much stronger condition than real differentiability: for instance, it guarantees a function f is *analytic* in a neighborhood of a point.

Theorem 2.2. *Suppose $f : U \to \mathbb{C}$ is holomorphic. Then, for all $p \in U$, there exists $\varepsilon > 0$ and unique complex coefficients $(a_k)_{0 \leq k < \infty}$ such that*

$$f(z) = \sum_{k=0}^{\infty} a_k (z-p)^k$$

for all z in the open disc $B(p, \varepsilon)$.

In particular, if f is complex differentiable, so too are all of its derivatives, so f is forced to be C^∞.

Analyticity endows holomorphic functions with an additional perk: they enjoy the following strong property of polynomial functions:

Theorem 2.3. *Let f be nonzero and holomorphic on U, and let $p \in U$ be a zero of f. Then there exists a unique integer $k \geq 1$ and a holomorphic function g on U with $g(p) \neq 0$ such that*

$$f(z) = (z-p)^k g(z).$$

While the results we have discussed thus far are already impressive, they are all local, i.e. they only concern the behavior of functions in a disc about a point. The following theorem shows that holomorphic functions also satisfy a remarkable global property: they are totally characterized by their values on a sufficiently clustered subset.

Theorem 2.4 (Identity theorem)**.** *Let f and g be holomorphic on a connected open set U. If $f \equiv g$ on some subset $V \subseteq U$ with a limit point in U, then $f \equiv g$ on U.*

Proof. It suffices to show $h := f - g \equiv 0$. Let $Z = \{z \in U : h^{(k)}(z) = 0 \text{ for all } k \geq 0\}$. Since V has a limit point $p \in U$ and h is continuous, $h(p) = 0$, implying that $h^{(k)}(p) = 0$ for all $k \geq 1$ as well. Then, applying the same argument to each derivative $h^{(k)}$, we have $p \in Z$, so Z is nonempty. Moreover, being an intersection of closed sets, Z is closed. Finally, Z is also open, since at any $q \in Z$, the Taylor series of h centered at q has all zero coefficients, so h vanishes on an entire disc around q. Now, by definition, a nonempty subset of a connected open set U that is simultaneously open and closed must be all of U, so $Z = U$, implying $h \equiv 0$ as desired. □

Indeed, the theory of holomorphic functions is so rigid that it extends to a broader class of functions, the so-called *meromorphic* functions.

Definition 2.5. A function f is *meromorphic* on U if it is holomorphic outside a discrete set $S \subset U$ and has a pole at each point of S.

By the identity theorem, both the zeros and poles of a meromorphic function are discrete. This gives meromorphic functions a relatively simple structure: they are holomorphic everywhere except at a discrete set of poles, and are largely characterized by their zeros and poles by Theorem 2.3.

The importance of this structure is best illustrated by the residue theorem, which shows that the integral of a meromorphic function over a closed contour is totally determined by local data at the poles it encloses.

The most important special case of the residue theorem is *Cauchy's integral formula*, which will be a key tool in proving Hartogs's phenomenon.

Theorem 2.6 (Cauchy's integral formula)**.** *Let f be holomorphic on an open set U containing a closed disc $\overline{B}(p,r) = \{z \in \mathbb{C} : |z-p| \le r\}$. Then, for all $z \in B(p,r)$,*

$$f(z) = \frac{1}{2\pi i} \oint_{|\omega - p| = r} \frac{f(\omega)}{\omega - z}\, d\omega.$$

That is, f is determined entirely by its values on the boundary of its domain, much like how fixing a temperature at the boundary of a disc determines the temperature at every interior point.[2]

2.2. **Functions of Several Complex Variables.**

We now move to the case of functions in $n \ge 2$ complex variables.

Definition 2.7. Let $U \subseteq \mathbb{C}^n$ be an open set. A function $f : U \to \mathbb{C}$ is said to be *holomorphic* on U if it is holomorphic in each variable separately. That is, for each j and each fixed choice of the remaining variables, the function $z_j \mapsto f(z_1, \dots, z_n)$ is holomorphic in the sense of Definition 2.1.

[2] This isn't just an analogy. In fact, every holomorphic function can be viewed in precisely this way by virtue of the fact that its real and imaginary parts are *harmonic*.

An equivalent but less obvious characterization is that f is holomorphic if and only if it is locally equal to an absolutely convergent power series in $z_1, \ldots, z_n$. That holomorphicity in each variable implies this much stronger joint condition is a nontrivial result known as Hartogs's theorem (not to be confused by the topic of this note!). As in one variable, we say f is *meromorphic* on U if it is holomorphic outside a closed set $S \subset U$ of complex codimension one and has poles along S.

Many results from one variable generalize to the case of n variables by straightforward induction on n. The identity theorem, for instance, extends directly: if $f, g : U \to \mathbb{C}$ are holomorphic on a connected open set $U \subseteq \mathbb{C}^n$ and agree on a nonempty open subset, then $f \equiv g$ on all of U. Together with Cauchy's integral formula, this is all we will need to prove Hartogs's phenomenon in the case that the singular locus K is finite.

3. Proof of the theorem

Over time, Hartogs's phenomenon has been investigated from various angles, leading to many different proofs. In most modern formulations, the theorem is proved using the theory of partial differential equations; see e.g. [4]. In this section, we prove the theorem in the special case that K is discrete and compact, i.e. where K is finite. We closely follow Hartogs's original proof, which uses Cauchy's integral formula and the identity theorem in tandem to handle the local and global cases respectively.

Theorem 3.1. *Let $U \subseteq \mathbb{C}^n$ be open with $n \geq 2$, and let $K \subset U$ be finite. Then, if f is holomorphic on $U \setminus K$, it extends uniquely to a holomorphic function on U.*

By the identity theorem, uniqueness is immediate, and by induction on $|K|$, it suffices to treat the case $K = \{p\}$. Without loss of generality, we fix $p = 0$.

Proof. Let f be holomorphic on $B \setminus \{0\}$, where $B \subset \mathbb{C}^n$ is an open ball around the origin. Write coordinates (z, w) with $z \in \mathbb{C}$ and $w \in \mathbb{C}^{n-1}$. For fixed small $w \neq 0$, the function $z \mapsto f(z, w)$ is holomorphic in z on a punctured disc, so it admits a Laurent expansion

$$f(z,w) = \sum_{k=-\infty}^{\infty} a_k(w)\, z^k, \qquad a_k(w) = \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z,w)}{z^{k+1}}\, dz,$$

where each $a_k(w)$ is holomorphic in w for $w \neq 0$ by differentiating under the integral sign. Then, $K = \{0\}$, so $K \cap (\mathbb{C} \times \{w\}) = \emptyset$ for all $w \neq 0$. That is, $z \mapsto f(z, w)$ is holomorphic on the full disc $\{|z| < r\}$. Its Laurent series therefore has no negative terms, i.e. $a_k(w) = 0$ for all $k < 0$ and all $w \neq 0$. Then, by the identity theorem applied to any component of w, $a_k \equiv 0$ on Δ_w, thus its Laurent series reduces to a Taylor series

$$f(z,w) = \sum_{k=0}^{\infty} a_k(w)\, z^k.$$

This defines a holomorphic extension of f to all of B. □

References

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[4] J. Lebl, *Tasty Bits of Several Complex Variables: A Whirlwind Tour of the Subject*, version 4.4, 2026. Available at `https://www.jirka.org/scv/scv.pdf`. 5