ALMOST INTEGERS

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1. INTRODUCTION

Number theory is distinguished among the mathematical subjects for its abundance of seemingly simple phenomena which have strikingly deep explanations. One interesting example is the *almost integers*, which, informally, are real numbers which differ from an integer by only a small margin. For example, consider the constant

$$e^{\pi\sqrt{163}} = 262537412640768743.99999999999925,$$

which was discovered in 1859 by Hermite and later studied by Ramanujan. Looking at the left-hand side, it is far from obvious that this expression might approximate an integer: not only is $e^{\pi\sqrt{163}}$ irrational, it is in fact transcendental.

In this note, we will explain the number theory behind Ramanujan's constant, and describe two other almost integers which arise in a similar fashion.

2. HEEGNER NUMBERS

We begin with a presumably unrelated observation due to Euler. Consider

$$f(n) = n^2 - n + 41. \tag{1}$$

By direct computation, one can verify that $f(m)$ is prime for all $0 \leq m \leq 40$. That is, f produces 41 consecutive prime numbers!

But how is this connected to Ramanujan's constant? Well, if we compute the discriminant of f, we see that

$$\Delta f = (-1)^2 - 4(1)(41) = -163.$$

As it turns out, this reappearance of 163 isn't just coincidence. Indeed, Euler's prime generating function $f(n)$ and Ramanujan's constant are both explained by the fact that 163 is one of nine so-called *Heegner numbers*.

We briefly recap some algebraic number theory. First recall that for any squarefree integer $D \in \mathbb{Z}$, the set

$$\mathbb{Q}(\sqrt{D}) = \{a + b\sqrt{D} \mid a, b \in \mathbb{Q}\}$$

is a field, analogous to an extended version of $\mathbb{Q}$. Likewise, one can define the *ring of integers* $\mathcal{O}_K$ of $K = \mathbb{Q}(\sqrt{D})$, which are to $\mathbb{Q}(\sqrt{D})$ as the ordinary integers $\mathbb{Z}$ are to $\mathbb{Q}$. If D is of the form $4k+1$, then $\mathcal{O}_{\mathbb{Q}(\sqrt{D})} = \mathbb{Z}\left[\frac{1+\sqrt{D}}{2}\right]$, else $\mathcal{O}_{\mathbb{Q}(\sqrt{D})} = \mathbb{Z}[\sqrt{D}] = \{a+b\sqrt{D} \mid a, b \in \mathbb{Z}\}$.

If the elements of $\mathcal{O}_K$ can be uniquely decomposed into the product of "primes," we say that $\mathcal{O}_K$ is a *unique factorization domain* (UFD). Note that $\mathcal{O}_K$ is not a UFD in general: for instance, in the ring of integers $\mathbb{Z}[\sqrt{-5}]$, we can factor 6 as the product of primes in two ways:

$$6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5}).$$

The below theorem tells us that the special examples of D such that $\mathcal{O}_{\mathbb{Q}(\sqrt{D})}$ is a UFD precisely give rise to prime-generating polynomials.

Theorem 2.1 (Rabinowitsch, [2]). *For a positive integer m, the polynomial*

$$x^2 - x + m$$

is prime for $0 \le x \le m-2$ if and only if $D := 4m-1$ is squarefree and the ring of integers

$$\mathbb{Z}\left[\frac{1+\sqrt{-D}}{2}\right] = \left\{a + b \cdot \frac{1+\sqrt{-D}}{2} \;\middle|\; a, b \in \mathbb{Z}\right\}$$

of $\mathbb{Q}(\sqrt{-D})$ is a unique factorization domain.

Note that the above theorem coupled with Euler's observation tells us that $D = -163$ has this property. Moreover, by the following theorem, there are exactly eight further examples.

Theorem 2.2 (Stark-Heegner, [4]). *The ring of integers of $\mathbb{Q}(\sqrt{-D})$ is a UFD if and only if*

$$D \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}.$$

These are the so-called *Heegner numbers*. Restricting to only those of the form $4m-1$, we have only seven examples:

$$3, 7, 11, 19, 43, 67, 163.$$

Along with 163, we may wonder, do the other numbers on this list also give rise to almost-integers? First, starting from the least-to-greatest, we can calculate

$$e^{\pi\sqrt{3}} \approx 230.765, \quad e^{\pi\sqrt{7}} \approx 4017.932, \quad e^{\pi\sqrt{11}} \approx 33506.14.$$

So far, it seems like we've hit a dead end: none of these are almost integers! However, if we check the larger Heegner numbers, something interesting happens:

$$e^{\pi\sqrt{43}} = 884736743.9998,$$
$$e^{\pi\sqrt{67}} = 147197952743.999998,$$
$$e^{\pi\sqrt{163}} = 262537412640768743.99999999999925.$$

In particular, it seems that if h is a sufficiently large Heegner number, $e^{\pi\sqrt{h}}$ approaches an integer. Strangely, each of the above examples also seem to approach integers ending in 744, which again is nonobvious without any prior calculation.

In the next section, we will explain this phenomenon via the *j-invariant*, a highly symmetric function which encodes a large amount of arithmetic data.

3. THE j-INVARIANT

In this section, we introduce the *j-invariant*, a complex-analytic function whose properties fully explain why $e^{\pi\sqrt{43}}$, $e^{\pi\sqrt{67}}$, and $e^{\pi\sqrt{163}}$ are almost integers. But first, we will require some complex analysis.

3.1. **Review of complex analysis.** Recall that the complex numbers $\mathbb{C}$ are expressions of the form $a + bi$, where $a, b \in \mathbb{R}$ and $i^2 = -1$. One can directly show that $\mathbb{C}$ is a field, and moreover it is *complete*, i.e. there is a sensible notion of calculus on $\mathbb{C}$.

By identifying complex numbers $a + bi$ with points (a, b), we can geometrically identify the set of complex numbers $\mathbb{C}$ with the plane $\mathbb{R}^2$, where the x-axis $\mathbb{R}$ represents the real line, and the y-axis $i\mathbb{R}$ represents the *imaginary line*. This also allows us to geometrically interpret subsets of $\mathbb{C}$, such as the *upper half-plane* $\mathbb{H} := \{a + bi \mid a, b \in \mathbb{R},\ b > 0\}$.

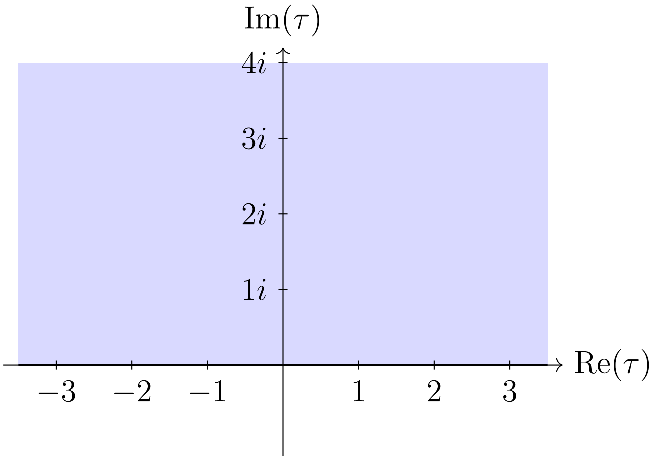

FIGURE 1. The complex upper half-plane $\mathbb{H}$.

Just as one studies real-differentiable functions in calculus, in complex analysis, one studies complex-differentiable functions, or *holomorphic functions*. One such function is the *complex exponential*, which can be defined by its Taylor series

$$\exp(\tau) := \sum_{n=0}^{\infty} \frac{\tau^n}{n!}$$

for a complex variable τ.[1] Apart from being holomorphic on $\mathbb{C}$, $\exp(\tau)$ satisfies another special property: it is $2\pi i$*-periodic*, namely $\exp(\tau + 2\pi i) = \exp(\tau)$. That is, $\exp(2\pi i\tau)$ is 1-periodic, so

$$\exp(2\pi i(\tau + 1)) = \exp(2\pi i\tau)$$

Indeed, $\exp(2\pi i\tau)$ is the fundamental 1-periodic function in a precise sense: every 1-periodic holomorphic function f can be expanded as a series[2] in $e^{2\pi i\tau}$ as follows:

$$f(\tau) = \sum_{n=-m}^{\infty} c_n q^n, \qquad q = e^{2\pi i\tau}.$$

This expression is called the *Fourier series* of f, and its coefficients are said to be its *Fourier coefficients*. This expansion is often quite useful, and in the case of the j-invariant, it precisely explains the existence of almost integers such as $e^{\pi\sqrt{163}}$.

3.2. **The j-invariant.** Having reviewed some complex analysis, it is finally time to define the j-invariant $j : \mathbb{H} \to \mathbb{C}$. In one formulation, it can be written as follows:

$$j(\tau) = \frac{\left(1 + 240\sum_{m\geq 1}\sum_{n\geq 0} m^3 q^{mn}\right)^3}{q(1-q)^{24}(1-q^2)^{24}\cdots}, \qquad q = e^{2\pi i\tau}.$$

At first glance, it is far from obvious that this rather strange expression has any meaningful interpretation.

However, as one can show, $j(\tau)$ is actually holomorphic on the upper half-plane and satisfies two fundamental relations:

$$j(\tau + 1) = j(\tau), \qquad j\left(-\tfrac{1}{\tau}\right) = j(\tau). \tag{2}$$

Moreover, every other holomorphic function on $\mathbb{H}$ satisfying the above equations can be expressed in terms of j. That is, $j(\tau)$ is the "simplest" 1-periodic holomorphic function f satisfying $f(-1/\tau) = f(\tau)$.

While the additional relation $j(-1/\tau) = j(\tau)$ may seem like only a slight improvement over 1-periodicity, note that it in fact imposes an infinite number of additional periods of $j(\tau)$ by composing the relations in (2). For instance,

$$j(\tau) = j\left(-\frac{1}{\tau}\right) = j\left(\frac{\tau - 1}{\tau}\right) = j\left(\frac{1}{1-\tau}\right) = j\left(\frac{3\tau - 2}{5\tau - 3}\right) = \cdots.$$

In fancier terms, (2) guarantees that $j(\tau)$ is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d},$$

so $j(\tau)$ is a so-called *modular function*.

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[1] We denote the complex exponential by e^τ or $\exp(\tau)$ interchangeably depending on the context.

[2] We also require that f satisfies some subtle growth conditions.

This invariance can be visually depicted by symmetries in the graph of j:

FIGURE 2. Graph of the j-invariant on the upper half-plane.

Note this graph repeats itself on vertical strips, and even exhibits a fractal structure. That is, it is infinitely self-similar on increasingly small scales.

This abundance of symmetries turns out to impose strong conditions on the Fourier coefficients of j, which can be written as follows:

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots, \qquad q = e^{2\pi i\tau}. \tag{3}$$

It is a surprising fact that the Fourier coefficients of $j(\tau)$ are all integers. Moreover, the number 744 reappears as the constant term.

The $\mathrm{SL}_2(\mathbb{Z})$-invariance of $j(\tau)$ also endows it with a great deal of number-theoretic significance. Again, in fancy terms, it allows $j(\tau)$ to be viewed as a holomorphic function on the modular curve $X(1)$, which parametrizes isomorphism classes of *elliptic curves*. This deep connection links the j-invariant to algebraic number theory, and gives rise to the following remarkable result:

Theorem 3.1. *Suppose that h is a Heegner number, so $h \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$. Then, we have that $j(\tau_h)$ is an integer, where*

$$\tau_h = \frac{1 + i\sqrt{h}}{2}.$$

In particular, for $h = 163$, we have the exact equality

$$j(\tau_{163}) = -262537412680768000. \tag{4}$$

Note that if we substitute τ_{163} into $q = e^{2\pi i\tau}$ we obtain

$$q = -e^{-\pi\sqrt{163}} \approx -3.81 \cdot 10^{-18},$$

which is an extremely small number which shrinks as we take powers of it. Therefore, plugging $e^{2\pi i\tau_{163}}$ into the Fourier expansion (3) of $j(\tau)$, we obtain

$$j(\tau_{163}) = -e^{\pi\sqrt{163}} + 744 + \text{tiny error terms}.$$

That is, rearranging and plugging in (4),

$$e^{\pi\sqrt{163}} = 262537412680768000 + 744 + \text{tiny error terms}.$$

For the second and third-largest Heegner numbers 43 and 67, we have by similar reasoning that

$$e^{\pi\sqrt{43}} = 884736000 + 744 + \text{tiny error terms},$$
$$e^{\pi\sqrt{67}} = 147197952000 + 744 + \text{tiny error terms}.$$

This simultaneously explains two mysterious observations: firstly, that the Heegner numbers give rise to almost integers, and secondly, these almost integers seem to nearly end in 744. Moreover, it tells us why the smaller Heegner numbers break this pattern: for those, the error terms are not quite small enough to be negligible.

This apparently innocent observation lies at only the start of the deep and intricate theory of *modular forms*, of which the j-invariant is only a first example. Most notably, these functions were a key tools in Wiles' proof of Fermat's Last Theorem, and are more generally of high number-theoretic interest. They have also found applications in conformal field theory from physics, where $\mathrm{SL}_2(\mathbb{Z})$-invariant physical systems be described in terms of elliptic functions and modular forms [5].

The Fourier coefficients of such functions also often encode extremely interesting information. For instance, in a phenomenon known as *monstrous moonshine*, the terms in the Fourier expansion of $j(\tau)$ were observed to coincide with representations of the so-called *Monster group*, an object which arises in an entirely different area of mathematics! For more on this fascinating topic, we refer the reader to [6, 7].

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